Exam 1 Review

General Differentiation Formulas

\[
\begin{align*}
(u + v)' &= u' + v' \\
(cu)' &= cu' \\
(uv)' &= u'v + uv' \quad \text{(product rule)} \\
\left(\frac{u}{v}\right)' &= \frac{u'v - uv'}{v^2} \quad \text{(quotient rule)} \\
\frac{d}{dx} f(u(x)) &= f'(u(x)) \cdot u'(x) \quad \text{(chain rule)}
\end{align*}
\]

You can remember the quotient rule by rewriting

\[
\left(\frac{u}{v}\right)' = (uv^{-1})'
\]

and applying the product rule and chain rule.

Implicit differentiation

Let’s say you want to find \(y'\) from an equation like

\[y^3 + 3xy^2 = 8\]

Instead of solving for \(y\) and then taking its derivative, just take \(\frac{d}{dx}\) of the whole thing. In this example,

\[
\begin{align*}
3y^2y' + 6xyy' + 3y^2 &= 0 \\
(3y^2 + 6xy)y' &= -3y^2 \\
y' &= \frac{-3y^2}{3y^2 + 6xy}
\end{align*}
\]

Note that this formula for \(y'\) involves both \(x\) and \(y\). Implicit differentiation can be very useful for taking the derivatives of inverse functions.

For instance,

\[y = \sin^{-1} x \Rightarrow \sin y = x\]

Implicit differentiation yields

\[(\cos y)y' = 1\]

and

\[y' = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}\]
Specific differentiation formulas

You will be responsible for knowing formulas for the derivatives and how to deduce these formulas from previous information: $x^n$, $\sin^{-1} x$, $\tan^{-1} x$, $\sin x$, $\cos x$, $\tan x$, $\sec x$, $e^x$, $\ln x$.

For example, let’s calculate $\frac{d}{dx} \sec x$:

$$\frac{d}{dx} \sec x = \frac{d}{dx} \frac{1}{\cos x} = -\frac{\sin x}{\cos^2 x} = \tan x \sec x$$

You may be asked to find $\frac{d}{dx} \sin x$ or $\frac{d}{dx} \cos x$, using the following information:

$$\lim_{h \to 0} \frac{\sin(h)}{h} = 1$$
$$\lim_{h \to 0} \frac{\cos(h) - 1}{h} = 0$$

Remember the definition of the derivative:

$$\frac{d}{dx} f(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Tying up a loose end

How to find $\frac{d}{dx} x^r$, where $r$ is a real (but not necessarily rational) number? All we have done so far is the case of rational numbers, using implicit differentiation. We can do this two ways:

1st method: base $e$

$$x = e^{\ln x}$$
$$x^r = (e^{\ln x})^r = e^{r \ln x}$$
$$\frac{d}{dx} x^r = \frac{d}{dx} e^{r \ln x} = e^{r \ln x} \frac{d}{dx} (r \ln x) = e^{r \ln x} \frac{r}{x}$$
$$\frac{d}{dx} x^r = x^r \left( \frac{r}{x} \right) = r x^{r-1}$$

2nd method: logarithmic differentiation

$$\frac{d}{dx} (\ln f)' = \frac{f'}{f}$$
$$f = x^r$$
$$\ln f = r \ln x$$
$$(\ln f)' = \frac{r}{x}$$
$$f' = f (\ln f)' = x^r \left( \frac{r}{x} \right) = r x^{r-1}$$
Finally, in the first lecture I promised you that you’d learn to differentiate anything—even something as complicated as $\frac{d}{dx} e^x \tan^{-1} x$

So let’s do it!

$$\frac{d}{dx} e^{uv} = e^{uv} \frac{d}{dx} (uv) = e^{uv} (u'v + uv')$$

Substituting,

$$\frac{d}{dx} e^{x \tan^{-1} x} = e^{x \tan^{-1} x} \left( \tan^{-1} x + x \left( \frac{1}{1 + x^2} \right) \right)$$